マチン(Machin)の公式

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

【導出方法】

$\tan\alpha = \dfrac{1}{5}$ と置く。三角関数の倍角の公式を用いると、

$$\begin{aligned}
\tan 2\alpha &= \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{\frac{2}{5}}{1-\left(\frac{1}{5}\right)^2} = \frac{10}{12} = \frac{5}{12} \\
\tan 4\alpha &= = \frac{2\tan 2\alpha}{1-\tan^2 2\alpha} = \frac{2\times\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2} = \frac{2\times 5\times 12}{144-25} = \frac{120}{119} \simeq 1
\end{aligned}$$

そこで、

$$\tan\left[\frac{\pi}{4} - 4\alpha\right] = \frac{1}{\tan\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - 4\alpha\right)\right]} = \frac{1}{\tan\left[\frac{\pi}{4} + 4\alpha\right]}$$

を求めてみよう。右辺分母は、

$$\begin{aligned}
\tan\left[\frac{\pi}{4} + 4\alpha\right] &= \frac{\sin(\pi/4+4\alpha)}{\cos(\pi/4+4\alpha)} = \frac{\sin(\pi/4)\cos(4\alpha)+\cos(\pi/4)\sin(4\alpha)}{\cos(\pi/4)\cos(4\alpha)-\sin(\pi/4)\sin(4\alpha)} = \frac{1+\tan(4\alpha)}{1-\tan(4\alpha)} \\
&= \frac{119+120}{119-120} = -239
\end{aligned}$$

となるので、

$$\tan\left[\frac{\pi}{4} - 4\alpha\right] = -\frac{1}{239}$$

すなわち、

$$\frac{\pi}{4} - 4\alpha = -\arctan\frac{1}{239}$$

ところで、$\alpha = \arctan\dfrac{1}{5}$ なので、

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

を得る。