Sums of Continuous Uniform Random Variables

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Let $X = x_i (i = 0,\ 1,\ 2, \cdots,\ n)$ be mutually independent random variables with the continuous uniform density $f(X)$ on the interval $0 \le x_i \le 1$.

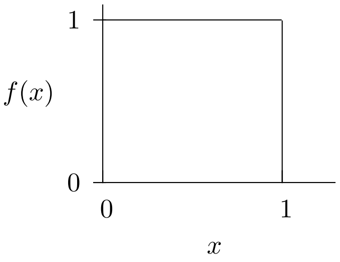

図 1: Probability density function $f(x)$

We seek the probability density function of $z = x_1 + x_2 + \cdots + x_n$.

1 The case $n = 2$

$$f_2(z) = \int_0^1 dx_2 \int_0^1 \delta(z - x_1 - x_2)\, dx_1 \tag{1}$$

First integrate with respect to x_1, and then with respect to x_2. Since the integration range of $x_1 = z - x_2$ is $0 \le x_1 \le 1$, we have $0 \le z - x_2 \le 1$. Therefore, $z - 1 \le x_2 \le z$. Comparing this with $0 \le x_2 \le 1$ gives

$$\max\{0,\ z-1\} \le x_2 \le \min\{1,\ z\}.$$

Thus, when $0 \le z \le 1$, $0 \le x_2 \le z$, and when $1 \le z \le 2$, $z - 1 \le x_2 \le 1$. Hence,

$$f_2(z) = \begin{cases} \displaystyle\int_0^z dx_2 = z & 0 \le z \le 1 \\ \displaystyle\int_{z-1}^1 dx_2 = 2 - z & 1 \le z \le 2 \end{cases} \tag{2}$$

is obtained.

2 The case $n = 3$

Put $z = x_1 + x_2 + x_3$; then $0 \le z \le 3$.

$$f_3(z) = \int_0^1 dx_3 \int_0^1 dx_2 \int_0^1 \delta(z - x_1 - x_2 - x_3)\, dx_1 \tag{3}$$

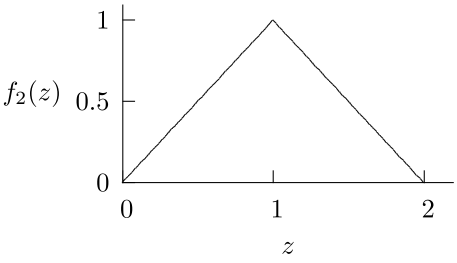

図 2: Probability density function $f_2(z)$

First integrate with respect to x_1, and then with respect to x_2. Since the integration range of $x_1 = z - x_2 - x_3$ is $0 \le x_1 \le 1$, we have $0 \le z - x_2 - x_3 \le 1$. Therefore, $z - 1 - x_3 \le x_2 \le z - x_3$. On the other hand, since $0 \le x_2 \le 1$,

$$\max\{z - 1 - x_3,\ 0\} \le x_2 \le \min\{z - x_3,\ 1\}.$$

That is,

(A) when $x_3 \le z - 1$, then $z - 1 - x_3 \le x_2 \le 1$;

(B) when $z - 1 \le x_3$, then $0 \le x_2 \le z - x_3$.

For the integration region to exist, we must have $z - 1 - x_3 \le 1$ and $0 \le z - x_3$; that is, $z - 2 \le x_3 \le z$. Comparing this with $0 \le x_3 \le 1$ gives

$$\max\{z - 2,\ 0\} \le x_3 \le \min\{z,\ 1\}.$$

Also, when $z \le 1$, there is no x_3 satisfying (A). When $1 \le z - 1$, there is no x_3 satisfying (B).

The Case of $0 \le z \le 1$

Since $z - 1 \le x_3$ and $0 \le x_3 \le z$, we have $0 \le x_3 \le z$.

$$f_3(z) = \int_0^z dx_3 \int_0^{z-x_3} dx_2 = \frac{z^2}{2} \tag{4a}$$

The Case of $1 \le z \le 2$

Since $0 \le x_3 \le 1$,

$$f_3(z) = \int_0^{z-1} dx_3 \int_{z-1-x_3}^1 dx_2 + \int_{z-1}^1 dx_3 \int_0^{z-x_3} dx_2 = -z^2 + 3z - \frac{3}{2} \tag{4b}$$

The Case of $2 \le z \le 3$

Since $x_3 \le z - 1$ and $z - 2 \le x_3 \le 1$, we have $z - 2 \le x_3 \le 1$.

$$f_3(z) = \int_{z-2}^1 dx_3 \int_{z-1-x_3}^1 dx_2 = \frac{1}{2}(z - 3)^2 \tag{4c}$$

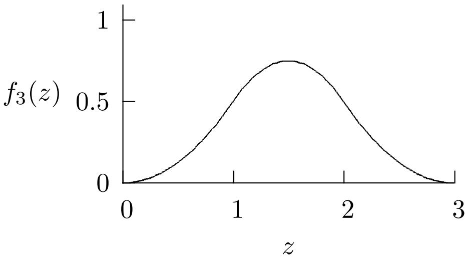

図 3: Probability density function $f_3(z)$

3 The case $n = 4$

Using equation (2), we have

$$\begin{aligned} f_4(z) &= \int_0^2 dy \int_0^2 \delta(z-x-y) f_2(x) f_2(y)\, dx \\ &= \int_0^2 f_2(z-y) f_2(y)\, dy \qquad (5) \end{aligned}$$

Since $z = x + y$ with $0 \leq x \leq 2,\ 0 \leq y \leq 2$, it is clear that $0 \leq z \leq 4$. Moreover, from $0 \leq y \leq 2$ we have $0 \leq z - x \leq 2$, namely $z - 2 \leq y \leq z$. Comparing this with $0 \leq y \leq 2$ gives

$$\max\{z-2,\ 0\} \leq y \leq \min\{z,\ 2\}.$$

When $0 \leq z \leq 2$, then $0 \leq y \leq z$, so

$$f_4(z) = \int_0^z f_2(z-y) f_2(y)\, dy \qquad (6)$$

When $2 \leq z \leq 4$, then $z - 2 \leq y \leq 2$, so

$$f_4(z) = \int_{z-2}^2 f_2(z-y) f_2(y)\, dy \qquad (7)$$

Equation (6) is further divided into two cases: $0 \leq z \leq 1$ and $1 \leq z \leq 2$.

The Case of $0 \leq z \leq 1$

$$f_4(z) = \int_0^z (z-y) y\, dy = \frac{z^3}{6} \qquad (8)$$

The Case of $1 \leq z \leq 2$

$$\begin{aligned} f_4(z) &= \int_0^{z-1} (2-z+y) y\, dy + \int_{z-1}^1 (z-y) y\, dy + \int_1^z (z-y)(2-y)\, dy \qquad (9) \\ &= -\frac{1}{2} z^3 + 2z^2 - 2z + \frac{2}{3} \qquad (10) \end{aligned}$$

Equation (7) is further divided into two cases: $2 \leq z \leq 3$ and $3 \leq z \leq 4$.

The Case of $2 \le z \le 3$

$$\begin{aligned} f_4(z) &= \int_{z-2}^{1} (2-z+y)y\,dy + \int_{1}^{z-1} (2-z+y)(2-y)\,dy + \int_{z-1}^{2} (z-y)(2-y)\,dy \\ &= \frac{1}{2}z^3 - 4z^2 + 10z - \frac{22}{3} \end{aligned}$$

The Case of $3 \le z \le 4$

$$f_4(z) = \int_{z-2}^{2} (2-z+y)(2-y)\,dy = \frac{1}{6}(4-z)^3 \tag{11}$$

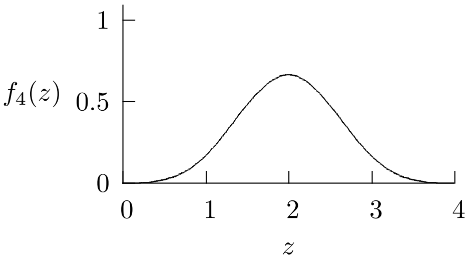

図 4: Probability density function $f_4(z)$

4 Central limit theorem

Let X be a random variable with the uniform density $f(X)$. To make its mean $\mu = \frac{1}{2}$ equal to zero, perform the transformation $Y = X - \frac{1}{2}$. We seek, for large n, the probability distribution of the arithmetic mean $\bar{Y} = (Y_1 + Y_2 + \cdots + Y_n)/n$ of the n independent random variables $Y_1,\ Y_2,\ \cdots,\ Y_n$:

$$f_{\bar{Y}}(\bar{y}) = \int_{-1/2}^{1/2} dy_1 \int_{-1/2}^{1/2} dy_2 \cdots \int_{-1/2}^{1/2} dy_n\, \delta\left(\bar{y} - \frac{y_1 + y_2 + \cdots + y_n}{n}\right) \tag{12}$$

First, substituting the integral representation of the delta function,

$$\delta\left(\bar{y} - \frac{y_1 + y_2 + \cdots + y_n}{n}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(\bar{y} - (y_1 + y_2 + \cdots + y_n)/n)}\,dk,$$

into equation (12) gives

$$\begin{aligned} f_{\bar{Y}}(\bar{y}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik\bar{y}} \prod_{i}^{n} \left(\int_{-1/2}^{1/2} e^{-iky_i/n}\,dy_i\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik\bar{y}} \left(\frac{\sin\frac{k}{2n}}{\frac{k}{2n}}\right)^n \end{aligned}$$

Now, $\left(\frac{\sin\frac{k}{2n}}{\frac{k}{2n}}\right)^n$ is an even function of k which, when n is sufficiently large, has a sharp peak equal to 1 near $k/2n \approx 0$. Near $k/2n \approx 0$,

$$\left(\frac{\sin\frac{k}{2n}}{\frac{k}{2n}}\right)^n \simeq 1 - \frac{k^2}{24n}$$

and therefore, approximating it by the exponential function $\exp\left(-\frac{k^2}{24n}\right)$, we obtain

$$\begin{aligned}
f_{\bar{Y}}(\bar{y}) &\simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp\left[-\frac{k^2}{24n} + ik\bar{y}\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp\left[-\frac{1}{24n}(k - 12in\bar{y})^2 - 6n\bar{y}^2\right] \\
&= \frac{1}{2\pi} e^{-6n\bar{y}^2}\int_{-\infty}^{\infty} dk \exp\left[-\frac{1}{24n}(k - 12in\bar{y})^2\right]
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_{-\infty}^{\infty} dk \exp\left[-\frac{1}{24n}(k - 12in\bar{y})^2\right] \\
&= \int_{-\infty}^{\infty} dk' \exp\left[-\frac{k'^2}{24n}\right] \\
&= \sqrt{24\pi n}
\end{aligned}$$

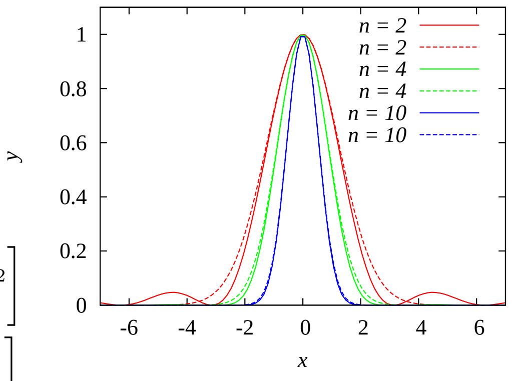

図 5: Graph of $y = (\sin x/x)^n$ (solid line) and graph of $y = e^{-nx^2/6}$ (dotted line)

and hence

$$f_{\bar{Y}}(\bar{y}) \simeq \sqrt{\frac{6n}{\pi}}\, e^{-6n\bar{y}^2} \tag{13}$$

holds. This is a normal distribution with variance $\sigma_n^2 = \frac{1}{12n}$. The variance of the uniform distribution is

$$\sigma^2 = \int_{-1/2}^{1/2} x^2\, dx = \frac{1}{12},$$

so

$$\sigma_n^2 = \frac{\sigma^2}{n} \tag{14}$$

holds, and equation (13) can be written as

$$f_{\bar{Y}}(\bar{y}) \simeq \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\bar{y}^2/2\sigma_n^2} \tag{15}$$

where σ_n^2 is given by equation (14).