

Derivation of the t -Distribution

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Student's t -distribution was introduced in 1908 by William Sealy Goset. The statistic variable t is defined by

$$t = \frac{u}{\sqrt{v/n}},$$

where u is a variable of the standard normal distribution $g(u)$, and v be a variable of the χ^2 distribution $T_n(v)$ of of the n degrees of freedom. Thus, we can express the distribution function of t in terms of $g(u)$ and $T_n(v)$.

The distribution function $g(u)$ and $T_n(v)$ are represented by

$$g(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \quad (1)$$

and

$$T_n(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{(n-2)/2} e^{-v/2} \quad (2)$$

respectively.

The probabilistic function $f_n(t)$ of $t = u/\sqrt{v/n}$ can then be written as

$$f_n(t) = \int \delta\left(t - \frac{u}{\sqrt{v/n}}\right) g(u) T_n(v) du dv,$$

where $\delta\left(t - u/\sqrt{v/n}\right)$ is a Dirac's delta function.

Before integrate over u , we introduce a variable $y = u/\sqrt{v/n}$, and integrate over y instead of u .

$$\begin{aligned} f_n(t) &= \int \sqrt{\frac{v}{n}} \delta(t - y) g(\sqrt{v/n} y) T_n(v) dy dv \\ &= \int \sqrt{\frac{v}{n}} g(\sqrt{v/n} t) T_n(v) dv \end{aligned}$$

Now, substituting eq.(1) and eq.(2) into this equation, we get

$$f_n(t) = \frac{1}{\sqrt{2\pi n} 2^{n/2} \Gamma(n/2)} \int_0^\infty v^{(n-1)/2} e^{-(1+t^2/n)v/2} dv$$

Furthermore, we rewrite the equation using a variable $x = \left(1 + \frac{t^2}{n}\right) \frac{v}{2}$ instead of v , to find that

$$f_n(t) = \frac{(1 + t^2/n)^{-(n+1)/2}}{\sqrt{\pi n} \Gamma(n/2)} \int_0^\infty x^{(n+1)/2-1} e^{-x} dx.$$

The integrand of the right hand side can be represented by using the gamma function,

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx.$$

Thus we obtain

$$f_n(t) = \frac{\Gamma(n+1)/2}{\sqrt{\pi n} \Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$

Furthermore, gamma functions are represented by beta function as

$$B(1/2, n/2) = \frac{\Gamma(1/2) \Gamma(n/2)}{\Gamma((n+1)/2)} = \frac{\sqrt{\pi} \Gamma(n/2)}{\Gamma((n+1)/2)}.$$

So we finally obtain the student's t-distribution function as follows,

$$f_n(t) = \frac{1}{\sqrt{n} B(1/2, n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$

We show the t distribution function as Figure 1. For comparison, we also show the standard normal distribution function ($N(0, 1)$ - solid line).

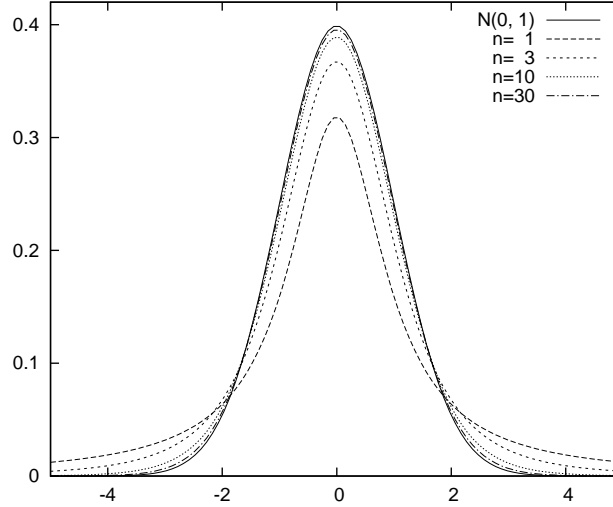


Figure 1: t distribution function $y = f_n(t)$ of $n = 1, 3, 10, 30$ degrees of freedom, and the standard normal distribution function $N(0, 1)$ as solid line.