Probability Density Functions of Sums, Products, Quotients, and Powers of Random Variables (PDFs of $X+Y,\ XY,\ X/Y,$ and X^Y)

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X and Y are mutually independent continuous random variables, and their probability density functions (PDFs) are given by $f(x)$ and $g(y)$, respectively. We derive the probability density functions of (a) $Z = X+Y$, (b) $Z = XY$, (c) $Z = X/Y$, and (d) $Z = X^Y$.

(a) $Z = X + Y$

$$\begin{aligned} p(z) &= \int \delta(z-x-y) f(x) g(y)\, dx\, dy \\ &= \int f(z-y) g(y)\, dy \end{aligned} \tag{1}$$

【Example 1】When the random variables x and y follow the uniform distribution on $0 \le x,\ y \le 1$,

The probability density functions are $f(x) = 1\ (0 \le x \le 1)$ and $g(y) = 1\ (0 \le y \le 1)$. The integration over x simply removes the delta function.

$$p(z) = \int_0^1 dy \int_0^1 \delta(z-x-y)\, dx$$

The range of the subsequent y-integration is clear from the line $y = -x+z$ drawn in the sample space of Fig. 1: when $0 \le z \le 1$, we have $0 \le y \le z$, and when $1 \le z \le 2$, we have $z-1 \le y \le 1$.

$$p(z) = \begin{cases} \displaystyle\int_0^z dy = z & 0 \leqq z \leqq 1 \\ \displaystyle\int_{z-1}^1 dy = 2-z & 1 \leqq z \leqq 2 \end{cases}$$

【Example 2】 For exponential random variables x and y,
$f_X(x) = \lambda e^{-\lambda x} (x \geqq 0),\ g_Y(y) = \lambda e^{-\lambda y} (y \geqq 0)$,

$$\begin{aligned} f_G(z) &= \lambda^2 \int \delta(z-x-y) e^{-\lambda x} e^{-\lambda y}\, dx dy \\ &= \lambda^2 e^{-\lambda z} \int_0^z dy \qquad (x = z-y \geqq 0,\ y \geqq 0\ \rightarrow\ 0 \leqq y \leqq z) \\ &= \lambda^2 z e^{-\lambda z} \end{aligned}$$

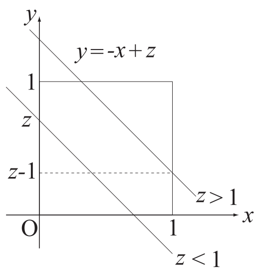

Fig. 1: Sample space and $y = -x + z$

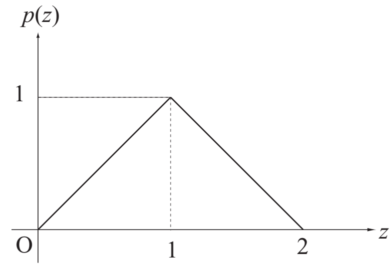

Fig. 2: Probability density function $p(z)$

The gamma distribution $\Gamma(\alpha, \lambda)$ is defined by

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

The exponential distribution is the gamma distribution with $\alpha = 1$. The sum of mutually independent exponentially distributed random variables follows the gamma distribution with $\alpha = 2$.

【Example 3】 For standard normal random variables x and y,
$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $g(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$,

$$\begin{aligned} f_N(z) &= \frac{1}{2\pi} \int \delta(z - x - y) e^{-(x^2+y^2)/2}\, dxdy \\ &= \frac{1}{2\pi} \int e^{-\left((z-y)^2+y^2\right)/2}\, dy \\ &= \frac{1}{2\pi} e^{-z^2/4} \int e^{-(y-z/2)^2}\, dy \end{aligned}$$

Let $t = y - z/2$. Then

$$\int e^{-(y-z/2)^2}\, dy = \int e^{-t^2}\, dt = \sqrt{\pi}$$

Therefore,

$$f_N(z) = \frac{1}{2\sqrt{\pi}} e^{-z^2/4}$$

This is the normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 2$.

(b) $Z = XY$

$$q(z) = \int \delta(z - xy) f(x) g(y)\, dx\, dy \tag{2}$$

To evaluate this integral, we use the following property of the δ function.

$$\delta\left(u(x)\right) = \frac{1}{|u'(\alpha)|} \delta(x - \alpha) \tag{3}$$

Here, α is a root of $u(x) = 0$. In this case, we have $u(x) = z - xy$, and $u'(x) = -y$. Substituting equation (3) into equation (2), we obtain

$$q(z) = \int \frac{1}{|y|} f(z/y) g(y)\, dy$$

as desired.

【Example 1】When the random variables x and y follow the uniform distribution on $0 \leq x,\ y \leq 1$,

$$q(z) = \int_z^1 \frac{dy}{y} = -\ln z$$

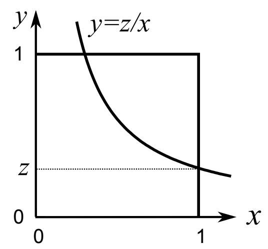

Fig. 3: Sample space and $y = z/x$

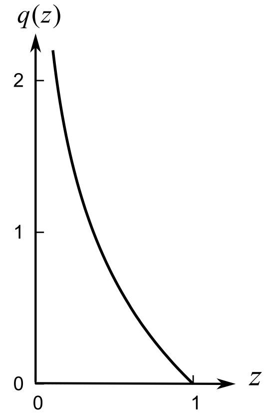

Fig. 4: Probability density function $q(z)$

【Example 2】Example 2 When the random variables x and y follow the exponential distribution,

$$\begin{aligned} f(z) &= \lambda^2 \int \delta(z - xy)\, e^{-\lambda x} e^{-\lambda y}\, dxdy \\ &= \lambda^2 \int_0^\infty \frac{1}{y}\, e^{-\lambda(z/y+y)}\, dy \end{aligned}$$

Let $t = \lambda y$. Then

$$f(z) = \lambda^2 \int_0^\infty \frac{1}{t}\, e^{-(\lambda^2 z/t + t)}\, dt \tag{4}$$

Now consider the function

$$K_n(z) = \frac{1}{2}\left(\frac{z}{2}\right)^n \int_0^\infty \exp\left(-t - \frac{z^2}{4t}\right) \cdot t^{-n-1}\, dt$$

which is called the modified Bessel function of the second kind. In particular, when $n = 0$,

$$K_0(z) = \frac{1}{2}\int_0^\infty \exp\left(-t - \frac{z^2}{4t}\right) \cdot t^{-1}\, dt \tag{5}$$

so, using equation (5), equation (4) can be rewritten as

$$f(z) = 2\lambda^2 K_0\left(2\lambda\sqrt{z}\right)$$

which gives the result.

【Example 3】 When the random variables x and y follow the standard normal distribution ,

$$f_Z(z) = \frac{1}{2\pi}\int_{-\infty}^\infty dy \int_{-\infty}^\infty \delta(z - xy) e^{-(x^2+y^2)/2}\, dx$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|y|}\,e^{-(z^2/y^2+y^2)/2}\,dy \\
&= \frac{1}{2\pi}\left\{-\int_{-\infty}^{0}\frac{1}{y}\,e^{-(z^2/y^2+y^2)/2}\,dy+\int_{0}^{\infty}\frac{1}{y}\,e^{-(z^2/y^2+y^2)/2}\,dy\right\} \\
&= \frac{1}{\pi}\int_{0}^{\infty}\frac{1}{y}\,e^{-(z^2/y^2+y^2)/2}\,dy
\end{aligned}$$

Let $t=\dfrac{y^2}{2}$. Since $\dfrac{dy}{y}=\dfrac{dt}{2t}$,

$$\begin{aligned}
f_Z(z) &= \frac{1}{2\pi}\int_{0}^{\infty}\frac{e^{-t-z^2/4t}}{t}\,dt \\
&= \frac{1}{\pi}K_0(z)
\end{aligned}$$

Thus, the modified Bessel function of the second kind appears once again.

(c) $\boldsymbol{Z = X/Y}$

$$\begin{aligned}
r(z) &= \int \delta(z-x/y)f(x)g(y)\,dx\,dy \\
&= \int |y|f(zy)g(y)\,dy
\end{aligned}$$

Here, $u(x)=z-x/y$ and $u'(x)=-1/y$ in equation (3) which is used for the integration over x.
【Example 1】When the random variables x and y follow the uniform distribution on $0\le x,\ y\le 1$,

$$r(z)=\int_0^1 dy\int_0^1\delta(z-x/y)\,dx$$

The range of the subsequent y-integration is clear from the line $y=x/z$ drawn in the sample space of Fig. (5): when $0\le z\le 1$, we have $0\le y\le 1$, and when $1\le z$, we have $0\le y\le 1/z$.

$$r(z)=\begin{cases}\displaystyle\int_0^1 y\,dy=\left[\frac{y^2}{2}\right]_0^1=\frac{1}{2} & 0\leqq z\leqq 1\\[2ex] \displaystyle\int_0^{1/z} y\,dy=\left[\frac{y^2}{2}\right]_0^{1/z}=\frac{1}{2z^2} & 1\leqq z\end{cases}$$

【Example 2】 When the random variables x and y follow the exponential distribution,

$$\begin{aligned}
f(z) &= \lambda^2\int\delta(z-x/y)\,e^{-\lambda x}e^{-\lambda y}\,dxdy \\
&= \lambda^2\int_0^{\infty} y\,e^{-\lambda(z+1)y}\,dy
\end{aligned}$$

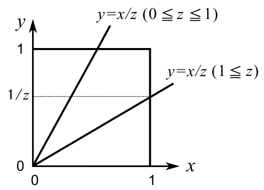

Fig. 5: Sample space and $y = x/z$

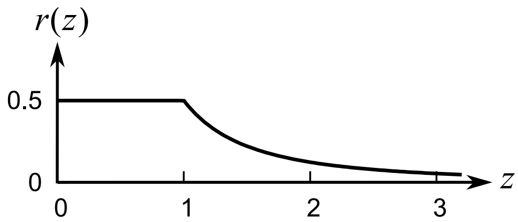

Fig. 6: Probability density function $r(z)$

Let $t = \lambda(z+1)y$. Then

$$\begin{aligned} f(z) &= \frac{1}{(z+1)^2}\int_0^\infty t\,e^{-t}\,dt \\ &= \frac{1}{(z+1)^2}\int_0^\infty t\left(-e^{-t}\right)'\,dt \\ &= \frac{1}{(z+1)^2}\left\{\Big[-t\,e^{-t}\Big]_0^\infty + \int_0^\infty e^{-t}\,dt\right\} \\ &= \frac{1}{(z+1)^2} \end{aligned}$$

That is,

$$f(z) = \frac{1}{(z+1)^2}$$

as desired.

【Example 3】 For standard normal random variables x and y,
$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $g(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}$,

$$\begin{aligned} f_C(z) &= \frac{1}{2\pi}\int_{-\infty}^\infty dy\int_{-\infty}^\infty \delta(z-x/y)e^{-(x^2+y^2)/2}\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty |y|\,e^{-(z^2+1)y^2/2}\,dy \\ &= \frac{1}{2\pi}\left\{-\int_{-\infty}^0 y\,e^{-(z^2+1)y^2/2}\,dy + \int_0^\infty y\,e^{-(z^2+1)y^2/2}\,dy\right\} \\ &= \frac{1}{\pi}\int_0^\infty y\,e^{-(z^2+1)y^2/2}\,dy \end{aligned}$$

Using $t = (1+z^2)y^2/2$ in place of the integration variable y,

$$\int_0^\infty y\,e^{-(z^2+1)y^2/2}\,dy = \frac{1}{1+z^2}\int_0^\infty e^{-t}\,dt = \frac{1}{1+z^2}$$

Therefore,

$$f_C(z) = \frac{1}{\pi(1+z^2)}$$

This is called the Cauchy distribution.

(d) $\boldsymbol{Z = X^Y}$

$$
\begin{aligned}
s(z) &= \int \delta(z - x^y) f(x) g(y)\, dx\, dy \\
&= \int \frac{1}{|y\, z^{1-1/y}|} f\left(z^{1/y}\right) g(y)\, dy
\end{aligned}
$$

Here, $u(x) = z - x^y$ and $u'(x) = -yx^{y-1}$ in equation (3) which is used for the integration over x.

【Example】Example For uniform random variables x and y on $0 \le x,\ y \le 1$,

$$
\begin{aligned}
s(z) &= \int_0^1 dy \int_0^1 \delta(z - x^y)\, dx \\
&= \int_0^1 \frac{1}{y\, z^{1-1/y}}\, dy \\
&= \frac{1}{z} \int_0^1 \frac{z^{1/y}}{y}\, dy \\
&= \frac{1}{z} \int_0^1 \frac{e^{(\ln z)/y}}{y}\, dy
\end{aligned}
$$

Let $t = \dfrac{\ln z}{y}$. Then

$$
e^{(\ln z)/y} = e^t, \qquad dt = -\ln z \frac{dy}{y^2} = -t \frac{dy}{y}
$$

Therefore,

$$
s(z) = -\frac{1}{z} \int_{-\infty}^{\ln z} \frac{e^t}{t}\, dt
$$

which gives the result.

The integral appearing here is called the exponential integral,

$$
Ei(\ln z) = \int_{-\infty}^{\ln z} \frac{e^t}{t}\, dt
$$

so it can be written as follows, and hence

$$
s(z) = -\frac{1}{z} Ei(\ln z), \qquad 0 \le z \le 1
$$

as desired.

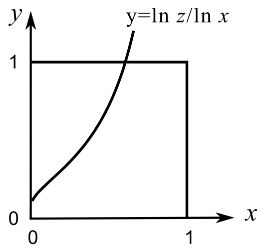

Fig. 7: Sample space and $y = \ln z / \ln x$

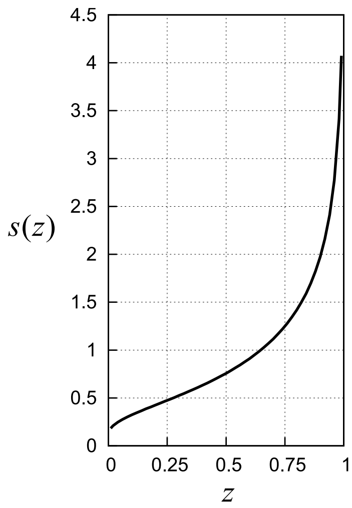

Fig. 8: Probability density function $s(z)$