Geometric and Exponential Distributions

Shoichi Midorikawa

Geometric Distribution

$$\mathrm{P}_G(x) = pq^{x-1}, \quad p+q=1, \quad x=1,2,3,\cdots$$

Exponential Distribution as a Continuous Limit of the Geometric Distribution

Let $t = \dfrac{x}{n}$ and consider the limit as $n \to \infty$. Then we seek $f(t)$ such that

$$f(t)\,\Delta t = \frac{1}{n} f(t) = pq^{nt-1}$$

Put $r = q^n$. Then

$$p = 1 - q = 1 - r^{1/n}$$

and hence

$$\frac{1}{n} f(t) = \left(1 - r^{1/n}\right) r^{t-1/n} \quad (1)$$
$$= \left(r^{-1/n} - 1\right) r^t \quad (2)$$

Since $r = \exp(\ln r)$,

$$r^{-1/n} = e^{-\frac{\ln r}{n}} \approx 1 - \frac{\ln r}{n}$$

therefore

$$f(t) = \lim_{n\to\infty} n\left(r^{-1/n} - 1\right) r^t = -\ln r\, e^{t \ln r}$$

Now put

$$\lambda = -\ln r$$

Then

$$f(t) = \lambda e^{-\lambda t}$$

is obtained. This is the exponential distribution.

From the Exponential Distribution to the Geometric Distribution

Starting from the exponential distribution

$$\mathrm{P}_E(x) = \lambda e^{-\lambda x}$$

we derive the geometric distribution $\mathrm{P}_G(n)$.

$$\begin{aligned} \mathrm{P}_G(n) &= \lambda \int_{n-1}^{n} e^{-\lambda t}\,dt \\ &= \left[-e^{-\lambda t}\right]_{n-1}^{n} \\ &= e^{-\lambda(n-1)} - e^{-\lambda n} \\ &= e^{-\lambda(n-1)}\left(1 - e^{-\lambda}\right) \end{aligned}$$

Here, put

$$p = 1 - e^{-\lambda}, \quad q = e^{-\lambda}$$

Then

$$\mathrm{P}_G(n) = pq^{n-1}$$

is obtained.

From the Geometric Distribution to the Exponential Distribution Again

Define the exponential distribution $\mathrm{P}_E(t) = f(t)$ in terms of the geometric distribution by

$$\mathrm{P}_G(n) = pq^{n-1} = q^{n-1} - q^n = \int_{n-1}^{n} f(t)\,dt$$

Differentiate both sides with respect to n.

For the left-hand side,

$$\frac{d}{dn} pq^{n-1} = p\frac{d}{dn}e^{(n-1)\ln q} = \ln q\ pq^{n-1} = -q^n \ln q + q^{n-1}\ln q$$

For the right-hand side,

$$\frac{d}{dn}\int_{n-1}^{n} f(t)\,dt = f(n) - f(n-1)$$

that is,

$$-q^n \ln q + q^{n-1}\ln q = f(n) - f(n-1)$$

therefore

$$f(n) = -q^n \ln q$$

holds.

Replacing n by t, we have

$$f(t) = -q^t \ln q$$

Furthermore, putting $\lambda = -\ln q$ gives $q = e^{-\lambda}$, and therefore

$$f(t) = \lambda e^{-\lambda t}$$

is obtained.

Mean and Variance

Mean and variance of the geometric distribution

$$\begin{aligned}\mu_G &= \sum_{x=1}^{\infty} xpq^{x-1} = \frac{1}{p} \\ \sigma_G^2 &= \sum_{x=1}^{\infty} x^2pq^{x-1} - \mu_G^2 = \left(\frac{2}{p^2} - \frac{1}{p}\right) - \frac{1}{p^2} = \frac{q}{p^2}\end{aligned}$$

Mean and variance of the exponential distribution

$$\begin{aligned}\mu_E &= \int_0^{\infty} tf(t)\,dt \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{\infty} xpq^{x-1} \\ &= \lim_{n\to\infty} \frac{1}{np} \\ &= \lim_{n\to\infty} \frac{1}{n(1-e^{-\lambda/n})}\end{aligned}$$

Here,

$$p = 1 - q = 1 - e^{-\lambda/n} \approx \frac{\lambda}{n}$$

and hence

$$\mu_E = \frac{1}{\lambda}$$

$$\begin{aligned}\int_0^{\infty} t^2 f(t)\,dt &= \lim_{n\to\infty} \frac{1}{n^2} \sum_{x=1}^{\infty} x^2pq^{x-1} \\ &= \lim_{n\to\infty} \frac{1}{n^2}\left(\frac{2}{p^2} - \frac{1}{p}\right) \\ &= \frac{2}{\lambda^2}\end{aligned}$$

therefore

$$\sigma_E^2 = \frac{1}{\lambda^2}$$