Gamma and Beta Distributions

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1 Derivation of the Gamma Distribution

Assume that the random variables $x_1,\ x_2,\ \cdots,\ x_n$ are mutually independent and each follows the exponential distribution $f(x_i) = \lambda e^{-\lambda x_i}\ \ (i = 1,\ 2,\ \cdots,\ n)$. Then the gamma distribution is derived as the distribution of the sum of these n independent variables, $z = x_1 + x_2 + \cdots + x_n$. That is,

$$f_n(z) = \lambda^n \int \delta(z - x_1 - x_2 - \cdots - x_n) e^{-\lambda(x_1+x_2+\cdots+x_n)}\, dx_1 dx_2 \cdots dx_n \tag{1}$$

This gives the required density.

The integral in Eq. (1) can be evaluated easily. Noting that $x_i \geq 0\ \ (i = 1,\ 2,\ \cdots,\ n)$, we obtain

$$f_n(z) = \lambda^n e^{-\lambda z} \int_0^z dx_n \int_0^{z-x_n} dx_{n-1} \cdots \int_0^{z-(x_4+\cdots+x_n)} dx_3 \int_0^{z-(x_3+x_4+\cdots+x_n)} dx_2 \tag{2}$$

The integration limits above are derived as follows.

Since $z = x_1 + x_2 + x_3 + \cdots + x_n$, we have $x_1 = z - (x_2 + x_3 + \cdots + x_n) \geq 0$, and therefore $x_2 \leq z - (x_3 + \cdots + x_n)$. Since $x_2 \geq 0$, it follows that $x_3 \leq z - (x_4 + \cdots + x_n)$. Likewise, because $x_3 \geq 0$, we obtain $x_4 \leq z - (x_5 + \cdots + x_n)$, and so on. Continuing this argument, we finally obtain $0 \leq x_n \leq z$ as the range of x_n.

Now evaluate the integral in Eq. (2) explicitly. First,

$$\int_0^{z-(x_3+x_4+\cdots+x_n)} dx_2 = (z - (x_4 + \cdots + x_n)) - x_3$$

Next,

$$\begin{aligned}\int_0^{z-(x_4+\cdots+x_n)} \{(z - (x_4 + \cdots + x_n)) - x_3\}\, dx_3 &= \frac{1}{2!}\left(z - (x_4 + \cdots + x_n)\right)^2 \\ &= \frac{1}{2!}\left((z - (x_5 + \cdots + x_n)) - x_4\right)^2\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{1}{2!}\int_0^{z-(x_5+\cdots+x_n)} \{(z - (x_5 + \cdots + x_n)) - x_4\}^2\, dx_4 &= \frac{1}{3!}\left(z - (x_5 + \cdots + x_n)\right)^3 \\ &= \frac{1}{3!}\left((z - (x_6 + \cdots + x_n)) - x_5\right)^3\end{aligned}$$

Repeating this procedure, we finally obtain

$$\begin{aligned}\frac{1}{(n-2)!}\int_0^z (z-x_n)^{n-2}\,dx_n &= \frac{1}{(n-1)!}\Big[-(z-x_n)^{n-1}\Big]_0^z \\ &= \frac{1}{(n-1)!}z^{n-1}\end{aligned}$$

Therefore,

$$f_n(z) = \frac{\lambda^n}{(n-1)!}z^{n-1}e^{-\lambda z}$$

Since $\Gamma(n) = (n-1)!$, this can be rewritten using the gamma function as

$$f_n(z) = \frac{\lambda^n}{\Gamma(n)}z^{n-1}e^{-\lambda z} \tag{3}$$

Now, the integral representation of the gamma function is

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\,dx$$

From the integral in Eq. (3), we obtain

$$\begin{aligned}\lambda^n\int_0^\infty z^{n-1}e^{-\lambda z}\,dz &= \int_0^\infty x^{n-1}e^{-x}\,dx \qquad (x=\lambda z) \\ &= \Gamma(n)\end{aligned}$$

Thus Eq. (3) is called the gamma distribution and is denoted by $Ga(n,\ \lambda)$.

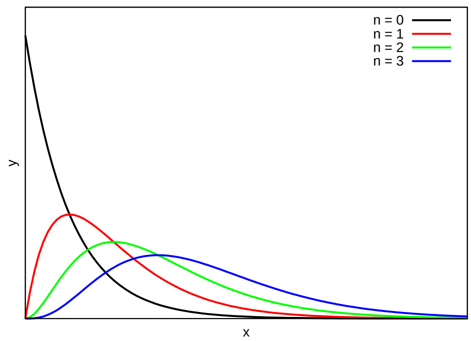

図 1: Gamma distribution $Ga(n,\ \lambda)$ for $n = 0,\ 1,\ 2,\ 3$

1.1 Mean and Variance of the Gamma Distribution

Mean

$$\begin{aligned}\mu &= \int_0^\infty z f_n(z)\,dz \\ &= \frac{\lambda^n}{\Gamma(n)}\int_0^\infty z^n e^{-\lambda z}\,dz\end{aligned}$$

Let $x = \lambda z$. Then

$$\begin{aligned}
\mu &= \frac{1}{\lambda\Gamma(n)} \int_0^\infty x^n e^{-x}\, dx \\
&= \frac{\Gamma(n+1)}{\lambda\Gamma(n)} \\
&= \frac{n}{\lambda}
\end{aligned}$$

Variance

$$\begin{aligned}
\sigma^2 &= \int_0^\infty z^2 f_n(z)\, dz - \mu^2 \\
&= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty z^{n+1} e^{-\lambda z}\, dz - \mu^2 \\
&= \frac{\Gamma(n+2)}{\lambda^2\Gamma(n)} - \frac{n^2}{\lambda^2} \\
&= \frac{n}{\lambda^2}
\end{aligned}$$

2 Moment Generating Function of the Gamma Distribution

Let us derive the moment generating function $M_Z(t)$ of the gamma distribution.

$$M_Z(t) = \int_0^\infty e^{tz} f_n(z)\, dz$$

Substituting Eq. (1) into this expression gives

$$M_Z(t) = \left(\lambda \int_0^\infty e^{(t-\lambda)x_1}\, dx_1\right)\left(\lambda \int_0^\infty e^{(t-\lambda)x_2}\, dx_2\right)\cdots\left(\lambda \int_0^\infty e^{(t-\lambda)x_n}\, dx_n\right)$$

Now, for $i = 1,\ 2,\ \cdots, n$,

$$\begin{aligned}
M_i(t) &= \left(\lambda \int_0^\infty e^{(t-\lambda)x_i}\, dx_i\right) \\
&= \frac{\lambda}{\lambda - t} \qquad \text{(provided that } t < \lambda)
\end{aligned}$$

Hence,

$$M_Z(t) = \left(\frac{\lambda}{\lambda - t}\right)^n \qquad \text{(provided that } t < \lambda)$$

3 Gamma Distribution and the χ^2 Distribution

The χ^2 distribution with n degrees of freedom is

$$T_n(z) = \frac{1}{2\Gamma(n/2)} \left(\frac{z}{2}\right)^{\frac{n}{2}-1} e^{-\frac{z}{2}}$$

Comparing this expression with Eq. (3), we see that the χ^2 distribution with n degrees of freedom is $Ga(n/2,\ 1/2)$.

4 Inverse Gamma Distribution

The inverse gamma distribution is obtained by replacing the variable z in Eq. (3) with $y = \dfrac{1}{z}$.

$$
\begin{aligned}
f_Y(y) &= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty \delta\left(y - \frac{1}{z}\right) z^{n-1} e^{-\lambda z}\, dz \\
&= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty z^2 \delta\left(z - \frac{1}{y}\right) z^{n-1} e^{-\lambda z}\, dz \\
&= \frac{\lambda^n}{\Gamma(n)} y^{-n-1} \exp\left(-\frac{\lambda}{y}\right)
\end{aligned}
$$

That is,

$$
f_Y(y) = \frac{\lambda^n}{\Gamma(n)} y^{-n-1} \exp\left(-\frac{\lambda}{y}\right)
$$

is called the inverse gamma distribution.

5 Beta Distribution

Consider two independent random variables x and y. Suppose that x follows the gamma distribution $Ga(m,\ \lambda)$ and y follows $Ga(n,\ \lambda)$. That is,

$$
f_m(x) = \frac{\lambda^m}{\Gamma(m)} x^{m-1} e^{-\lambda x} \tag{4a}
$$

$$
f_n(y) = \frac{\lambda^n}{\Gamma(n)} y^{n-1} e^{-\lambda y} \tag{4b}
$$

5.1 Distribution of $x/(x+y)$

The distribution of the random variable $z = \dfrac{x}{x+y}$ is given by

$$
f_Z(z) = \int \delta\left(z - \frac{x}{x+y}\right) f_m(x) f_n(y)\, dx dy
$$

To evaluate this integral, let us integrate with respect to y first. Since

$$
\delta\left(z - \frac{x}{x+y}\right) = \frac{x}{z^2} \delta\left(y - \frac{(1-z)x}{z}\right)
$$

we have

$$
f_Z(z) = \frac{1}{z^2} \int_0^\infty x\, f_m(x)\ f_n\left(\frac{1-z}{z} x\right) dx \tag{5}
$$

Substituting Eqs. (4a) and (4b) into this expression, we obtain

$$
\begin{aligned}
f_Z(z) &= \frac{\lambda^{m+n}}{\Gamma(m)\Gamma(n)} z^{-n-1} (1-z)^{n-1} \int_0^\infty x^{m+n-1} e^{-\lambda x/z}\, dx \\
&= \frac{1}{\Gamma(m)\Gamma(n)} z^{m-1} (1-z)^{n-1} \int_0^\infty t^{m+n-1} e^{-t}\, dt \qquad \left(t = \frac{\lambda}{z} x\right) \\
&= \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} z^{m-1} (1-z)^{n-1} \\
&= \frac{z^{m-1}(1-z)^{n-1}}{B(m,n)} \qquad (0 \le z \le 1)
\end{aligned}
$$

where

$$\Gamma(m+n) = \int_0^\infty t^{m+n-1} e^{-t}\,dt$$

and

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

were used.

The integral representation of the beta function $B(m,n)$ is

$$B(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1}\,dz$$

Therefore, the probability density function

$$f_Z(z) = \frac{z^{m-1}(1-z)^{n-1}}{B(m,n)} \qquad (0 \le z \le 1) \tag{6}$$

is called the beta distribution and is denoted by $Be(m,n)$.

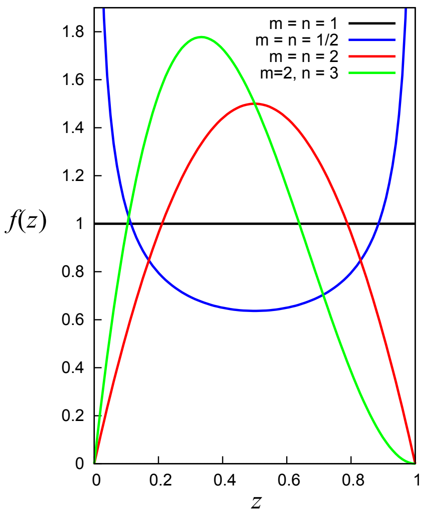

図 2: Beta distribution $Be(m,n)$

5.2 Mean and Variance of the Beta Distribution

Mean

$$\begin{aligned}
\mu &= \int_0^1 z f_Z(z)\,dz \\
&= \frac{1}{B(m,n)} \int_0^1 z^m (1-z)^{n-1}\,dz \\
&= \frac{B(m+1,n)}{B(m,n)} \\
&= \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \\
&= \frac{m}{m+n}
\end{aligned}$$

Variance

$$\begin{aligned}
\sigma^2 &= \int_0^1 z^2 f_Z(z)\,dz - \mu^2 \\
&= \frac{1}{B(m,n)} \int_0^1 z^{m+1}(1-z)^{n-1}\,dz - \mu^2 \\
&= \frac{B(m+2,n)}{B(m,n)} - \mu^2 \\
&= \frac{\Gamma(m+2)\Gamma(n)}{\Gamma(m+n+2)} \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} - \mu^2 \\
&= \frac{(m+1)m}{(m+n+1)(m+n)} - \frac{m^2}{(m+n)^2} \\
&= \frac{mn}{(m+n)^2(m+n+1)}
\end{aligned}$$

5.3 Distribution of x/y

The distribution of the random variable $u = \dfrac{x}{y}$ is given by

$$f_U(u) = \int \delta\left(u - \frac{x}{y}\right) f_m(x) f_n(y)\, dx dy$$

To evaluate this integral, this time we integrate with respect to x first. Since

$$\delta\left(u - \frac{x}{y}\right) = y\delta\left(x - uy\right)$$

we have

$$f_U(u) = \int y f_m(uy) f_n(y)\, dy \tag{7}$$

Substituting Eqs. (4a) and (4b) into this expression, we obtain

$$f_U(u) = \frac{\lambda^{m+n}}{\Gamma(m)\Gamma(n)} u^{m-1} \int_0^\infty y^{m+n-1} e^{-\lambda(1+u)y}\, dy \tag{8}$$

Let $t = \lambda(1+u)y$. Then

$$\begin{aligned} f_U(u) &= \frac{1}{\Gamma(m)\Gamma(n)} \frac{u^{m-1}}{(1+u)^{m+n}} \int_0^\infty t^{m+n-1} e^{-t}\, dt \\ &= \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \frac{u^{m-1}}{(1+u)^{m+n}} \\ &= \frac{1}{B(m,n)} \frac{u^{m-1}}{(1+u)^{m+n}} \end{aligned} \tag{9}$$

This is the desired result.

Now, the integral representation of the beta function is

$$B(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1}\, dz = \int_0^\infty \frac{u^{m-1}}{(1+u)^{m+n}}\, du \qquad u = \frac{z}{1-z}$$

Thus, Eq. (9) can also be regarded as a type of beta distribution.

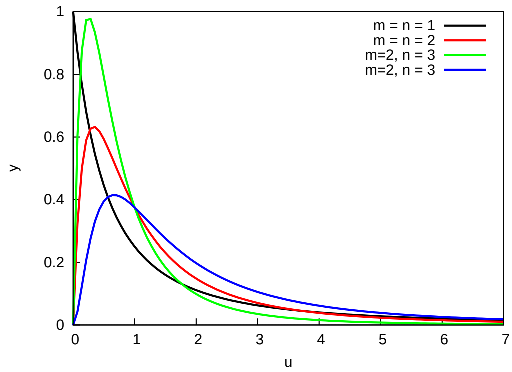

図 3: Distribution $f_U(u)$

The relation between $z^{m-1}(1-z)^{n-1}$ and $\dfrac{u^{m-1}}{(1+u)^{m+n}}$ is given by

$$z^{m-1}(1-z)^{n-1} = \int_0^\infty \delta\left(z - \frac{u}{1+u}\right) \frac{u^{m-1}}{(1+u)^{m+n}}\, du$$

This can be proved easily by using the well-known formula for the delta function,

$$\delta\left(g(u)\right) = \frac{1}{|g'(\alpha)|}\delta(u-\alpha) \qquad \text{where } \alpha \text{ is a solution of } g(u) = 0.$$

In the present case, $\delta\left(z - \dfrac{u}{1+u}\right) = \dfrac{1}{(1-z)^2}\delta\left(u - \dfrac{z}{1-z}\right)$.

Therefore, the relation between Eqs. (6) and (9) is

$$f_Z(z) = \frac{1}{(1-z)^2} f_U\left(\frac{z}{1-z}\right)$$