

# Cauchy Distribution

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## 1 Cauchy Distribution

Let  $x$  be a random variable with probability density function

$$f_X(x) = \frac{1}{\pi(1+x^2)}.$$

Then  $x$  is said to follow the standard Cauchy distribution. A distinctive feature of this distribution is that neither its expectation nor its variance exists.

The expectation

$$\mu = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x}{1+x^2} dx$$

might appear to be equal to 0 by symmetry, since  $xf_X(x)$  is an odd function. In fact, however,

$$\mu = \frac{1}{\pi} \int_0^{+\infty} \frac{x}{1+x^2} dx + \frac{1}{\pi} \int_{-\infty}^0 \frac{x}{1+x^2} dx = \infty - \infty,$$

so it is undefined.

Similarly,

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{x^2}{1+x^2} dx$$

diverges, and therefore the variance is also undefined.

The general form of the probability density function of the Cauchy distribution is given by

$$f_X(x) = \frac{\gamma}{\pi((x-x_0)^2 + \gamma^2)},$$

where  $x_0$  and  $\gamma$  are parameters.

In physics, this type of function arises as the solution describing damped forced oscillations in classical mechanics and resonances in quantum mechanics. For this reason, it is also known as the Lorentz distribution or the Breit–Wigner distribution.

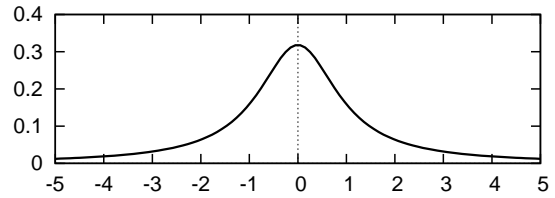


図 1: Cauchy distribution

## 2 Reproductive Property

Let  $i = 1, 2, 3, \dots, n$ , and suppose that the random variables  $x_i$  are mutually independent and identically distributed according to  $f(x_i) = \frac{1}{\pi(1+x_i^2)}$ . Then the distribution of  $z = x_1 + x_2 + \dots + x_n$  is given by

$$\begin{aligned}
 f_Z(z) &= \int_{-\infty}^{+\infty} \delta(z - x_1 - x_2 \cdots - x_n) f(x_1) f(x_2) \cdots f(x_n) dx_1 dx_2 \cdots dx_n \\
 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik(z-x_1-x_2 \cdots -x_n)} f(x_1) f(x_2) \cdots f(x_n) dx_1 dx_2 \cdots dx_n dk \\
 &= \frac{1}{2\pi^{n+1}} \int_{-\infty}^{+\infty} dk e^{ikz} \prod_{i=1}^n \int_{-\infty}^{+\infty} \frac{e^{-ikx_i}}{1+x_i^2} dx_i \\
 &= \frac{1}{2\pi^{n+1}} \int_{-\infty}^{+\infty} dk e^{ikz} \left[ \int_{-\infty}^{+\infty} \frac{e^{-ikx}}{1+x^2} dx \right]^n.
 \end{aligned}$$

Using the residue theorem from complex analysis, we obtain

$$\int_{-\infty}^{+\infty} \frac{e^{-ikx}}{1+x^2} dx = \int_{-\infty}^{+\infty} \frac{e^{-ikx}}{(x+i)(x-i)} dx = \begin{cases} \pi e^{-k} & k \geq 0 \\ \pi e^k & k < 0 \end{cases}$$

Hence,

$$\begin{aligned}
 f_Z(z) &= \frac{1}{2\pi} \left[ \int_0^{+\infty} e^{(iz-n)k} dk + \int_{-\infty}^0 e^{(iz+n)k} dk \right] \\
 &= \frac{1}{2\pi} \left[ -\frac{1}{iz-n} + \frac{1}{iz+n} \right] \\
 &= \frac{1}{\pi} \frac{n}{z^2 + n^2}.
 \end{aligned}$$

Next, consider the distribution of  $\bar{x} = z/n = (x_1 + x_2 + \dots + x_n)/n$ .

$$\begin{aligned}
 f_{\bar{X}}(\bar{x}) &= \int_{-\infty}^{+\infty} \delta\left(\bar{x} - \frac{z}{n}\right) f_Z(z) dz \\
 &= \frac{1}{\pi} \int_{-\infty}^{+\infty} \delta\left(\bar{x} - \frac{z}{n}\right) \frac{n}{z^2 + n^2} dz \\
 &= \frac{n}{\pi} \int_{-\infty}^{+\infty} \delta(n\bar{x} - z) \frac{n}{z^2 + n^2} dz \\
 &= \frac{1}{\pi(\bar{x}^2 + 1)}.
 \end{aligned}$$

Thus, the sample mean of mutually independent random variables having the same Cauchy distribution again follows the original Cauchy distribution.

### 3 Distribution of the Reciprocal

Suppose that the random variable  $x$  follows the Cauchy distribution

$$f_X(x) = \frac{1}{\pi(1+x^2)}.$$

We now determine the distribution of its reciprocal,  $y = 1/x$ .

$$f_Y(y) = \int_{-\infty}^{+\infty} \delta\left(y - \frac{1}{x}\right) \frac{1}{\pi(1+x^2)} dx.$$

Here we use the identity  $\delta(\varphi(x)) = \frac{1}{|\varphi'(x_0)|} \delta(x - x_0)$  (where  $x_0$  is a root of  $\varphi(x) = 0$ ). Then,

$\delta\left(y - \frac{1}{x}\right) = \frac{1}{y^2} \delta\left(x - \frac{1}{y}\right)$ . Therefore,

$$\begin{aligned} f_Y(y) &= \frac{1}{y^2} \int_{-\infty}^{+\infty} \delta\left(x - \frac{1}{y}\right) \frac{1}{\pi(1+x^2)} dx \\ &= \frac{1}{\pi(1+y^2)}. \end{aligned}$$

Hence, the reciprocal also follows a Cauchy distribution.

### 4 Uniform Distribution and Cauchy Distribution

Suppose that the random variable  $t$  is uniformly distributed on the interval  $-\frac{\pi}{2} < t < \frac{\pi}{2}$ :

$$f_T(t) = \frac{1}{\pi}.$$

Then the distribution  $f_X(x)$  of  $x = \tan t$  is given by

$$\begin{aligned} f_X(x) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \delta(x - \tan t) dt \\ &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \delta(t - \arctan(x)) \cos^2 t dt \\ &= \frac{1}{\pi(1+x^2)}. \end{aligned}$$

That is,  $x$  follows a Cauchy distribution.

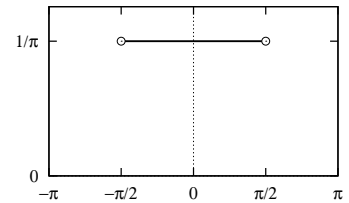


图 2: Uniform distribution