Weibull Distribution

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1 Survival Function

Let $\tau \geq 0$ be a random variable representing lifetime, and let its probability density function be $f(\tau)$. The function that represents the probability that the lifetime τ exceeds a given value t is called the survival function, and it is defined by

$$S(t) = \mathrm{P}(\tau > t) = \int_t^\infty f(\tau) d\tau \tag{1}$$

On the other hand, the cumulative distribution function $F(t)$ is given by

$$\begin{aligned} F(t) &= \int_0^t f(\tau) d\tau &(2) \\ &= 1 - S(t) &(3) \end{aligned}$$

Differentiating the cumulative distribution function $F(t)$ yields the probability density function $f(t)$:

$$\frac{dF(t)}{dt} = f(t) \tag{4}$$

The hazard function $h(t)$ is defined by

$$\begin{aligned} h(t) &= \lim_{\Delta\tau \to 0} \frac{P(t \leq \tau \leq t + \Delta\tau | \tau \geq t)}{\Delta\tau} \\ &= \frac{\displaystyle\lim_{\Delta\tau \to 0} \frac{P(t \leq \tau \leq t + \Delta\tau)}{\Delta\tau}}{P(\tau > t)} \\ &= \frac{\displaystyle\lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} \int_t^{t+\Delta\tau} f(t)\, d\tau}{\displaystyle\int_t^\infty f(\tau) d\tau} \end{aligned}$$

that is,

$$h(t) = \frac{f(t)}{\displaystyle\int_t^\infty f(\tau)\, d\tau} \tag{5}$$

From Eq. (1),

$$S(t) = \int_t^\infty f(\tau)\, d\tau, \qquad S'(t) = -f(t)$$

and therefore,

$$h(t) = -\frac{S'(t)}{S(t)}$$

is obtained. Replacing t by τ in this expression and rearranging, we obtain

$$\frac{dS(\tau)}{S(\tau)} = -h(\tau)d\tau$$

Integrating this equation from 0 to t, and using $S(0) = 1$, gives

$$S(t) = \exp\left\{-\int_0^t h(\tau)d\tau\right\}$$

We define

$$H(t) = \int_0^t h(\tau)d\tau$$

as the cumulative hazard function.

2 Exponential Distribution and Weibull Distribution

Exponential Distribution

The probability density function of the exponential distribution is

$$f(t) = \lambda e^{-\lambda t} \tag{6}$$

The survival function is

$$S(t) = \lambda \int_t^\infty e^{-\lambda\tau} d\tau = e^{-\lambda t}$$

The cumulative distribution function is

$$F(t) = 1 - e^{-\lambda t}$$

Substituting Eq. (6) into Eq. (5), we obtain

$$h(t) = \lambda$$

Also,

$$\frac{dF(t)}{dt} = \lambda e^{-\lambda t}$$

which reproduces the probability density function of the exponential distribution.

Weibull Distribution

For the exponential distribution $f(t) = \lambda e^{-\lambda t}$, let $t = x^n$, and consider the distribution $f_X(x)$ of the variable x.

$$f_X(x) = \lambda \int_0^\infty \delta\left(x - \sqrt[n]{t}\right) e^{-\lambda t}\, dt \tag{7}$$

To evaluate this integral, let $\varphi(t) = x - \sqrt[n]{t}$ and expand around $t = x^n$. Since $\varphi(x^n) = 0$,

$$\begin{aligned}\varphi(t) &= \varphi'(x^n)(t - x^n) + \cdots \\ &= -\frac{1}{n}x^{-(n-1)}(t - x^n) + \cdots\end{aligned}$$

Hence,

$$\delta\left(x - \sqrt[n]{t}\right) = nx^{n-1}\delta(t - x^n)$$

Substituting this into Eq. (1) and integrating with respect to t, we obtain

$$\begin{aligned} f_X(x) &= \lambda n x^{n-1}\int_0^\infty \delta(t - x^n)e^{-\lambda t}\,dt \\ &= \lambda n x^{n-1}e^{-\lambda x^n} \end{aligned} \tag{8}$$

This is called the Weibull distribution.

The general form of the Weibull distribution is obtained from Eq. (2) by setting $n = \alpha,\ \lambda = 1/\beta^\alpha$:

$$f_X(x) = \left(\frac{\alpha}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^\alpha} \tag{9}$$

From the cumulative distribution function

The cumulative distribution function $F(t)$ of the exponential distribution is

$$F(t) = 1 - e^{-\lambda t}$$

Now let $t = x^n$. Then

$$F_X(x) = 1 - e^{-\lambda x^n}$$

Differentiating this expression with respect to x,

$$\begin{aligned} \frac{dF_X(x)}{dx} &= \lambda n x^{n-1}e^{-\lambda x^n} \\ &= f_X(x) \end{aligned}$$

which gives the probability density function of the Weibull distribution, Eq. (8).

Mean

The mean of the Weibull distribution is obtained by substituting Eq. (9) into

$$\mu = \int_0^\infty x F_X(x)\,dx$$

Thus,

$$\mu = \left(\frac{\alpha}{\beta}\right)\int_0^\infty x\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^\alpha}\,dx$$

Now let $u = (x/\beta)^\alpha$. Then

$$\begin{aligned} x\left(\frac{\alpha}{\beta}\right)\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^\alpha} &= \alpha u\, e^{-u} \\ dx &= \frac{\beta}{\alpha}u^{\frac{1}{\alpha}-1}\,du \end{aligned}$$

Therefore,

$$\mu = \beta\int_0^\infty u^{\frac{1}{\alpha}}e^{-u}\,du$$

Using

$$\Gamma(n) = \int_0^\infty u^{n-1}e^{-u}\,du$$

we can write

$$\mu = \beta\Gamma\left(1+\frac{1}{\alpha}\right)$$

Variance

The variance is given by

$$\sigma^2 = \int_0^\infty x^2 f_X(x)\,dx - \mu^2$$

Here,

$$\begin{aligned}\int_0^\infty x^2 f_X(x)\,dx &= \alpha\beta\int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha+1} e^{-(x/\beta)^\alpha}\,dx \\ &= \beta^2\int_0^\infty u^{\frac{2}{\alpha}} e^{-u}\,du \\ &= \beta^2\Gamma\left(1+\frac{2}{\alpha}\right)\end{aligned}$$

Therefore,

$$\sigma^2 = \beta^2\left\{\Gamma\left(1+\frac{2}{\alpha}\right) - \Gamma^2\left(1+\frac{1}{\alpha}\right)\right\}$$

is obtained.

3 Distribution of a Ratio

Let x and y be random variables following Weibull distributions with the same value of α but different values of β:

$$f_1(x) = \left(\frac{\alpha}{\beta_1}\right)\left(\frac{x}{\beta_1}\right)^{\alpha-1} e^{-(x/\beta_1)^\alpha}, \quad f_2(y) = \left(\frac{\alpha}{\beta_2}\right)\left(\frac{y}{\beta_2}\right)^{\alpha-1} e^{-(y/\beta_2)^\alpha}$$

Consider the distribution of the ratio $z = x/y$ of these two random variables.

$$\begin{aligned}f_Z(z) &= \int_0^\infty \delta(z - x/y) f_1(x) f_2(y)\,dx\,dy \\ &= \int_0^\infty y\delta(x - yz) f_1(x) f_2(y)\,dx\,dy \\ &= \int_0^\infty y f_1(yz) f_2(y)\,dy \\ &= \frac{\alpha^2}{\beta_1\beta_2}\int_0^\infty y\left(\frac{yz}{\beta_1}\right)^{\alpha-1}\left(\frac{y}{\beta_2}\right)^{\alpha-1} e^{-(yz/\beta_1)^\alpha} e^{-(y/\beta_2)^\alpha}\,dy \qquad (10)\end{aligned}$$

Now let $u = (y/\beta_2)^\alpha$. Then Eq. (10) becomes

$$f_Z(z) = \alpha\left(\frac{\beta_2}{\beta_1}\right)\left(\frac{\beta_2}{\beta_1}z\right)^{\alpha-1}\int_0^\infty u\exp\left\{-\left[\left(\frac{\beta_2}{\beta_1}z\right)^\alpha + 1\right]u\right\}du \qquad (11)$$

Furthermore, letting $r = (\beta_2/\beta_1)^\alpha$ and $v = (rz^\alpha + 1)u$,

$$f_Z(z) = \alpha\frac{rz^{\alpha-1}}{(rz^\alpha+1)^2}\int_0^\infty ve^{-v}\,dv$$

Since

$$\int_0^\infty ve^{-v}\,dv = 1$$

it follows that

$$f_Z(z) = \frac{r\alpha z^{\alpha-1}}{(rz^\alpha+1)^2} \tag{12}$$

Furthermore, letting $w = rz^\alpha$,

$$\begin{aligned} f_W(w) &= \int_0^\infty \delta(w - rz^\alpha)\frac{\alpha z^{\alpha-1}}{(rz^\alpha+1)^2}\,dz \\ &= \frac{1}{(w+1)^2} \end{aligned}$$

This is an F distribution with degrees of freedom $(2,\ 2)$. That is, the random variable $w = \left(\frac{x/\beta_1}{y/\beta_2}\right)^\alpha$ follows an F distribution with degrees of freedom $(2,\ 2)$.