

Poisson Process and Gamma Distribution

Shoichi Midorikawa

1 Poisson Distribution

Suppose that the event S occurs, on average, λ times per unit time. To analyze this event, divide the unit time interval into n equal parts. In each small subinterval, the event S occurs at most once. If the probability of this is p , then its value is λ/n . The probability that the event S occurs k times in one unit of time is given by the binomial distribution:

$$\begin{aligned} P(k) &= {}_n C_k p^k (1-p)^{n-k} \\ &= \frac{n(n-1)\cdots(n-k+1)}{k!} p^k (1-p)^{n-k} \end{aligned}$$

Now consider the limit as $n \rightarrow \infty$. Since $np = \lambda$ is constant,

$$\begin{aligned} P(k) &= \frac{(1 - \frac{1}{n}) \cdots (1 - \frac{k-1}{n})}{k!} (np)^k \left\{ \left(1 - \frac{\lambda}{n}\right)^{n/\lambda} \right\}^\lambda \frac{1}{\left(1 - \frac{\lambda}{n}\right)^k} \\ &\xrightarrow{n \rightarrow \infty} \frac{\lambda^k e^{-\lambda}}{k!} \end{aligned}$$

Thus, define the new probability function

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!} \tag{1}$$

This is called the Poisson distribution.

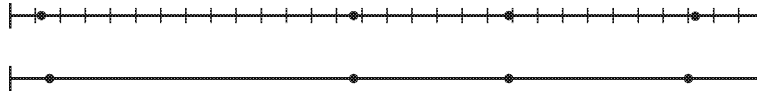


図1 Binomial distribution and Poisson distribution

2 Exponential Distribution

Next, let us find the probability that the event S occurs again t units of time after it has occurred.

When the unit time interval is divided into n parts, the probability that the number of small intervals between two events is x follows the geometric distribution and is given by

$$P(x) = p(1 - p)^x$$

Here, since $t = x/n$, $\Delta t = 1/n$, replacing $P(x)$ with $P(t)\Delta t$ gives

$$\begin{aligned} P(t)\Delta t &= \frac{\lambda}{n} \left\{ \left(1 - \frac{\lambda}{n} \right)^{n/\lambda} \right\}^{\lambda t} \\ &= \lambda \left\{ \left(1 - \frac{\lambda}{n} \right)^{n/\lambda} \right\}^{\lambda t} \Delta t \end{aligned}$$

Finally, taking the limit as $n \rightarrow \infty$ gives the exponential distribution

$$P(t) = \lambda e^{-\lambda t} \tag{2}$$

3 Poisson Process

In Section 1, we considered the probability that an event occurs k times in one unit of time. To generalize this and extend it to an arbitrary time interval $[0, t]$, replace λ by λt :

$$p_k(t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \tag{3}$$

When this is regarded as a stochastic process, it is called a Poisson process.

Transition Probabilities and Differential Equations

Suppose that the rare event S occurs, on average, λ times per unit time. Let $p_0(h)$ denote the probability that the event S does not occur during a small time interval h . Then the probability $1 - p_0(h)$ that it occurs at least once is λh if higher-order terms are ignored. That is,

$$1 - p_0(h) = \lambda h + o(h)$$

However, since S is a rare event, it may be said that the event S practically cannot occur two or more times during the short time interval h .

Now let h be sufficiently small, and suppose that the number of occurrences of the event S after time $t + h$ has elapsed is k . This means either that the event S occurred k times by time t and nothing occurred during the subsequent time interval h , or that the event S occurred $k - 1$ times by time t and one event occurred during the subsequent time interval h . Therefore,

$$p_k(t + h) = p_k(t)(1 - \lambda h) + p_{k-1}(t)\lambda h + o(h) \quad (k \geq 1)$$

Rewriting this gives

$$\frac{p_k(t+h) - p_k(t)}{h} = -\lambda p_k(t) + \lambda p_{k-1}(t)$$

Taking the limit as $h \rightarrow 0$ gives

$$p'_k(t) = -\lambda p_k(t) + \lambda p_{k-1}(t) \quad (k \geq 1) \quad (4)$$

When $k = 0$, $p_{-1}(0)$ is impossible, so

$$p'_0(t) = -\lambda p_0(t) \quad (5)$$

holds.

Next, let us solve the differential-difference equations (4) and (5). First, define the generating function

$$\begin{aligned} P(t, s) &= p_0(t) + p_1(t)s + p_2(t)s^2 + p_3(t)s^3 + \dots \\ &= \sum_{k=0}^{\infty} p_k(t)s^k \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} P(t, s) &= p'_0(t) + p'_1(t)s + p'_2(t)s^2 + p'_3(t)s^3 + \dots \\ &= -\lambda p_0(t) + [-\lambda p_1(t) + \lambda p_0(t)]s + [-\lambda p_2(t) + \lambda p_1(t)]s^2 + \dots \\ &= -\lambda [p_0(t) + p_1(t)s + p_2(t)s^2 + p_3(t)s^3 + \dots] \\ &\quad + \lambda s [p_0(t) + p_1(t)s + p_2(t)s^2 + p_3(t)s^3 + \dots] \\ &= -\lambda(1-s)P(t, s) \end{aligned}$$

That is,

$$\left(\frac{\partial}{\partial t} + \lambda(1-s) \right) P(t, s) = 0$$

This equation is easily solved as

$$\begin{aligned} P(t, s) &= e^{-\lambda(1-s)t} \\ &= e^{-\lambda t} e^{\lambda s t} \\ &= \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} s^k \end{aligned}$$

Therefore,

$$p_k(t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

is obtained.

4 Gamma Distribution

Let the random variables t_1, t_2, \dots, t_n be mutually independent and follow the same exponential distribution (2). Then the distribution followed by the random variable $t = t_1 + t_2 + \dots + t_n$ is given by

$$\begin{aligned} f_n(t) &= \lambda^n \int \delta(z - t_1 - t_2 - \dots - t_n) e^{-\lambda(t_1+t_2+\dots+t_n)} dt_1 dt_2 \dots dt_n \\ &= \lambda^n e^{-\lambda t} \int_0^t dt_n \int_0^{t-t_n} dt_{n-1} \dots \int_0^{t-(t_4+\dots+t_n)} dt_3 \int_0^{t-(t_3+t_4+\dots+t_n)} dt_2 \end{aligned}$$

This integral can be computed easily, yielding

$$f_n(t) = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} \quad (6)$$

This is called the gamma distribution. For details, see “[Gamma and Beta Distributions](#)”.

5 Poisson Process and Gamma Distribution

The gamma distribution $f_n(t) = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t}$ represents the probability density function for the time t until the event S has occurred n times. Therefore, the integral of the function $f_n(t)$ over $t > T$ is the probability that the time t until the event S occurs n times is longer than T . This is equal to the probability that the number of occurrences of the event S in the interval $[0, T]$ is less than n , and hence

$$\int_T^\infty \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} dt = \sum_{k=0}^{n-1} \frac{(\lambda T)^k e^{-\lambda T}}{k!}$$

holds. This can be easily verified by evaluating the integral on the left-hand side using integration by parts.

Indeed,

$$\begin{aligned} \frac{\lambda^n}{(n-1)!} \int_T^\infty t^{n-1} e^{-\lambda t} dt &= \frac{\lambda^n}{(n-1)!} \int_T^\infty t^{n-1} \left(-\frac{e^{-\lambda t}}{\lambda} \right)' dt \\ &= \frac{(\lambda T)^{n-1} e^{-\lambda T}}{(n-1)!} + \frac{\lambda^{n-1}}{(n-2)!} \int_T^\infty t^{n-2} e^{-\lambda t} dt \\ &= \frac{(\lambda T)^{n-1} e^{-\lambda T}}{(n-1)!} + \frac{(\lambda T)^{n-2} e^{-\lambda T}}{(n-2)!} + \dots + \frac{(\lambda T) e^{-\lambda T}}{1!} + \frac{\lambda}{0!} \int_T^\infty e^{-\lambda t} dt \\ &= \frac{(\lambda T)^{n-1} e^{-\lambda T}}{(n-1)!} + \frac{(\lambda T)^{n-2} e^{-\lambda T}}{(n-2)!} + \dots + \frac{(\lambda T) e^{-\lambda T}}{1!} + \frac{e^{-\lambda T}}{0!} \\ &= \sum_{k=0}^{n-1} \frac{(\lambda T)^k e^{-\lambda T}}{k!} \end{aligned}$$

is obtained.