The F-Distribution

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Let x_1 and x_2 are independent variables obeying the χ^2 distributions of the n_1 and n_2 degrees of freedom in each other. These distribution functions are given by

$$T_{n_i}(x_i) = \frac{1}{2^{n_i/2}\Gamma(n_i/2)} x_i^{n_i/2-1} e^{-x_i/2}, \tag{1}$$

where $i = 1$ and 2. Now let's consider the distribution of the variable x which is the ratio x_1/n_1 and x_2/n_2,

$$x = \frac{x_1/n_1}{x_2/n_2}.$$

We will write the distribution function of x as $f_{n_1,\,n_2}(x)$. Then it will be given by

$$f_{n_1,\,n_2}(x) = \int_0^\infty \delta\left(x - \frac{x_1/n_1}{x_2/n_2}\right) T_{n_1}(x_1) T_{n_2}(x_2)\, dx_1 dx_2, \tag{2}$$

where $\delta\left(x - \dfrac{x_1/n_1}{x_2/n_2}\right)$ is a Dirac's delta function. Before integrate over x_1, we will introduce a variable $y = n_2 x_1/n_1 x_2$, and use this variable intead of x_1.

$$f_{n_1,\,n_2}(x) = \int_0^\infty \frac{n_1}{n_2} x_2 \delta\,(x - y)\, T_{n_1}(n_1 x_2 y/n_2) T_{n_2}(x_2)\, dy dx_2$$

The integration over y is then simply to replace y by x.

$$f_{n_1,\,n_2}(x) = \int_0^\infty \frac{n_1}{n_2} x_2 T_{n_1}(n_1 x_2 x/n_2) T_{n_2}(x_2)\, dx_2 \tag{3}$$

Using the explicit form of $T_{n_1}(n_1 x_2 x/n_2)$ and $T_{n_2}(x_2)$ given by eq.(1), we can rewrite eq.(3) as

$$f_{n_1,\,n_2}(x) = \frac{x^{(n_1-2)/2}}{2^{(n_1+n_2)/2}\Gamma(n_1/2)\Gamma(n_2/2)} \left(\frac{n_1}{n_2}\right)^{n_1/2} \int_0^\infty x_2^{(n_1+n_2)/2-1} e^{-(1+n_1 x/n_2)x_2/2}\, dx_2$$

Furthermore, we change the integration variable x_2 for $t = (1 + n_1 x/n_2)x_2/2$. Then we find

$$f_{n_1,\,n_2}(x) = \frac{x^{(n_1-2)/2}}{\Gamma(n_1/2)\Gamma(n_2/2)(1 + n_1 x/n_2)^{(n_1+n_2)/2}} \left(\frac{n_1}{n_2}\right)^{n_1/2} \int_0^\infty t^{(n_1+n_2)/2-1} e^{-t}\, dt$$

The last integral is represented by the Gamma function,

$$\Gamma\left((n_1 + n_2)/2\right) = \int_0^\infty t^{(n_1+n_2)/2-1} e^{-t}\, dt.$$

We then get

$$f_{n_1,\,n_2}(x) = \frac{\Gamma\left((n_1+n_2)/2\right) x^{(n_1-2)/2}}{\Gamma(n_1/2)\Gamma(n_2/2)(1+n_1 x/n_2)^{(n_1+n_2)/2}} \left(\frac{n_1}{n_2}\right)^{n_1/2} \tag{4}$$

This equation called as Snedecor's F distribution or the Fisher-Snedecor distribution with $(n_1,\, n_2)$ degrees of freedom. By using the Beta function which is defined by

$$B(\alpha,\,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

we can rewrite the eq.(4) as

$$f_{n_1,\,n_2}(x) = \frac{1}{B(n_1/2,\,n_2/2)} \left(\frac{n_1}{n_2}\right)^{n_1/2} \frac{x^{(n_1-2)/2}}{(1+n_1 x/n_2)^{(n_1+n_2)/2}}.$$

We show in Fig.1, the F distributon functions with characteristic degrees of freedom.

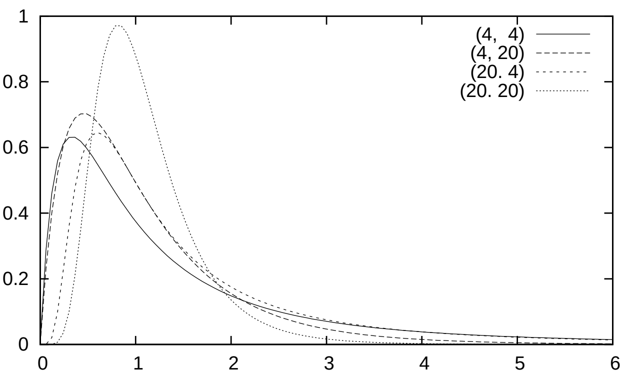

Figure 1: F distributions with $(n_1,\, n_2) = (4,\, 4),\quad (4,\, 20),\quad (20,\, 4),\, (20,\, 20)$ degrees of freedom.

Some Property of F distribution

What is the relation between the distribution of $x = (x_1/n_1)/(x_2/n_2)$ and that of $z = 1/x = (x_2/n_2)/(x_1/n_1)$? We want to change the variable x of $f_{n1,n2}(x)$ with z. We can realize this by the following integral.

$$f_{n_2,\,n_1}(z) = \int f_{n_1,\,n_2}(x)\delta\left(z - \frac{1}{x}\right)\, dx \tag{5}$$

Let $a(x) = z - \dfrac{1}{x}$, and expand $a(x)$ around $x = \dfrac{1}{z}$,

$$a(x) = z^2\left(x - \frac{1}{z}\right) + \cdots$$

We then get

$$\delta\left(a(x)\right) = \delta\left(z^2\left(x - \frac{1}{z}\right)\right) = z^{-2}\delta\left(x - \frac{1}{z}\right).$$

Substituting this int eq.(5) and integrating over x,

$$\begin{aligned}\int f_{n_1,\,n_2}(x)\delta\left(z - \frac{1}{x}\right)\,dx &= z^{-2}\int f_{n_1,\,n_2}(x)\delta\left(x - \frac{1}{z}\right)\,dx \\ &= z^{-2} f_{n_1,\,n_2}\left(\frac{1}{z}\right)\end{aligned}$$

we finally find the relation

$$f_{n_2,\,n_1}(z) = z^{-2} f_{n_1,\,n_2}\left(\frac{1}{z}\right).$$