問題　不定積分 $\displaystyle \int \frac{\mathbf{1}}{\boldsymbol{x^2+1}}\, \boldsymbol{dx}$ を求めよ

解答

$$
\begin{aligned}
\int \frac{1}{x^4+1}\, dx &= \int \frac{1}{(x^2-e^{\frac{\pi}{2}i})(x^2+e^{\frac{\pi}{2}i})}\, dx \\
&= -\frac{i}{2}\int \left\{ \frac{1}{x^2-e^{\frac{\pi}{2}i}} - \frac{1}{x^2+e^{\frac{\pi}{2}i}} \right\} dx \\
&= -\frac{i}{2}\int \left\{ \frac{1}{(x-e^{\frac{\pi}{4}i})(x+e^{\frac{\pi}{4}i})} - \frac{1}{(x-e^{-\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})} \right\} dx \\
&= -\frac{i}{2}\int \left\{ \frac{e^{-\frac{\pi}{4}i}}{2}\left[\frac{1}{x-e^{\frac{\pi}{4}i}} - \frac{1}{x+e^{\frac{\pi}{4}i}} \right] - \frac{e^{\frac{\pi}{4}i}}{2}\left[\frac{1}{x-e^{-\frac{\pi}{4}i}} - \frac{1}{x+e^{-\frac{\pi}{4}i}} \right] \right\} dx \\
&= -\frac{i}{2}\left\{ \frac{e^{-\frac{\pi}{4}i}}{2}\ln\left(\frac{x-e^{\frac{\pi}{4}i}}{x+e^{\frac{\pi}{4}i}} \right) - \frac{e^{\frac{\pi}{4}i}}{2}\ln\left(\frac{x-e^{-\frac{\pi}{4}i}}{x+e^{-\frac{\pi}{4}i}} \right) \right\} \\
&= -\frac{i}{2}\left\{ \frac{1}{2\sqrt{2}}\ln\left(\frac{(x-e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})}{(x+e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})} \right) - \frac{i}{2\sqrt{2}}\ln\left(\frac{(x-e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})}{(x+e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})} \right) \right\} \\
&= -\frac{i}{4\sqrt{2}}\ln\left(\frac{(x-e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})}{(x+e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})} \right) - \frac{1}{4\sqrt{2}}\ln\left(\frac{(x-e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})}{(x+e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})} \right)
\end{aligned}
$$

ここで、

$$
\begin{aligned}
\left(x-e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i} \right) &= (x^2-1)-\sqrt{2}\ ix \\
&= -i\sqrt{x^4+1}\left(\frac{\sqrt{2}\ x}{\sqrt{x^4+1}} + \frac{x^2-1}{\sqrt{x^4+1}}\ i \right) \\
&= -i\sqrt{x^4+1}\ e^{i\varphi}
\end{aligned}
$$

とおくと、

$$
\cos\varphi = \frac{\sqrt{2}\ x}{\sqrt{x^4+1}}, \quad \sin\varphi = \frac{x^2-1}{\sqrt{x^4+1}}
$$

すなわち、

$$
\tan\varphi = \frac{x^2-1}{\sqrt{2}\ x}, \qquad \text{または、}\varphi = \arctan\left(\frac{x^2-1}{\sqrt{2}\ x} \right)
$$

また、

$$
\left(x+e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i} \right) = i\sqrt{x^4+1}\ e^{-i\varphi}
$$

なので、

$$
\ln\left(\frac{(x-e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i})}{(x+e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i})} \right) = \ln\left(-e^{2i\varphi} \right) = \ln e^{2i\varphi+\pi i} = 2i\varphi+\pi i
$$

一方、

$$
\begin{aligned}
(x-e^{\frac{\pi}{4}i})(x-e^{-\frac{\pi}{4}i}) &= x^2-\sqrt{2}\ x+1 \\
(x+e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i}) &= x^2+\sqrt{2}\ x+1
\end{aligned}
$$

となるので、結局、

$$
\int \frac{1}{x^4+1}\, dx = \frac{1}{2\sqrt{2}}\arctan\left(\frac{x^2-1}{\sqrt{2}\ x} \right) - \frac{1}{4\sqrt{2}}\ln\left(\frac{x^2-\sqrt{2}\ x+1}{x^2+\sqrt{2}\ x+1} \right) + C
$$

を得る。

【注】

$\left(x-e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i}\right)$ は、また、

$$\begin{aligned}\left(x-e^{\frac{\pi}{4}i})(x+e^{-\frac{\pi}{4}i}\right) &= \sqrt{x^4+1}\left(\frac{x^2-1}{\sqrt{x^4+1}}-\frac{\sqrt{2}\ x}{\sqrt{x^4+1}}\ i\right) \\ &= \sqrt{x^4+1}\ e^{-i\theta}\end{aligned}$$

と書ける。ここで、

$$\cos\theta=\frac{x^2-1}{\sqrt{x^4+1}},\quad \sin\theta=\frac{\sqrt{2}\ x}{\sqrt{x^4+1}}$$

すなわち、

$$\tan\theta=\frac{\sqrt{2}\ x}{x^2-1},\qquad \text{または、}\qquad \varphi=\arctan\left(\frac{\sqrt{2}\ x}{x^2-1}\right)$$

そして、φ と θ の関係は、

$$\theta+\varphi=\frac{\pi}{2}$$

である。

そこで、この不定積分の解は、

$$\int\frac{1}{x^4+1}\,dx=-\frac{1}{2\sqrt{2}}\arctan\left(\frac{\sqrt{2}\ x}{x^2-1}\right)-\frac{1}{4\sqrt{2}}\ln\left(\frac{x^2-\sqrt{2}\ x+1}{x^2+\sqrt{2}\ x+1}\right)+C'$$

または、

$$\int\frac{1}{x^4+1}\,dx=\frac{1}{2\sqrt{2}}\arctan\left(\frac{\sqrt{2}\ x}{1-x^2}\right)+\frac{1}{4\sqrt{2}}\ln\left(\frac{x^2+\sqrt{2}\ x+1}{x^2-\sqrt{2}\ x+1}\right)+C$$

などと書くこともできる。